Centre of quantum group in roots of unity and the restriction of integrable models

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A Belavin

Landau Institute for Theoretical Physics, Chernogolovka 142432, Russia

E-mail: belavin@itp.ac.ru

Abstract
We show the connection between the extended centre of the quantum group
in roots of unity and the restriction of the *XXZ* model. We also give explicit
expressions for operators that respect the restriction and act on the state space of
the restricted models. The formulae for these operators are verified by explicit
calculation for third-degree roots; they are conjectured to hold in the general
case.

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1. Introduction

Alcaraz *et al* [1] discovered a remarkable fact: the *XXZ* model with the special open boundary
conditions (OBC) and a rational value of the anisotropy parameter admits a restriction. The
model arising as a result of the restriction coincides in the thermodynamic limit with one of
the minimal models of CFT. The algebraic reason for the restriction was explained in [2] and
[3]. It was shown in [2] that the *XXZ* model with the OBC considered in [1] has not only
integrability but also $U_q(sl(2))$ symmetry. In roots of unity, the state space of the model
decomposes to the sum of 'good' and 'bad' representations of $U_q(sl(2))$. The restriction of
Alcaraz *et al* is equivalent to throwing out 'bad' parts and keeping only the highest vectors
of 'good' representations. In [3], a new monodromy matrix was constructed that is bilinear
in terms of generators of the quantum group $A(u)$, $B(u)$, $C(u)$, $D(u)$ connected in the usual
way with the Hamiltonian of the model in [1] and compatible with integrability. The twisted
trace of this monodromy matrix (Sklyanin transfer matrix) also has $U_q(sl(2))$ symmetry and
admits the restriction.

　　In this paper, we generalize the Pasqier–Saleur construction. We show that not only
$U_q(sl(2))$-invariant but also a much wider class of OBC indeed admits the restriction (the
Sklyanin construction of the transfer matrix also works for this wider class of OBC).

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The decisive condition for compatibility of the Hamiltonian, the transfer matrix and other operators with the restriction is their 'weak' commutativity with a special element of the quantum group. The notion of 'weak' commutativity and its connection with the extension of the centre of the quantum group in roots of unity is explained below.

In section 2, we describe the conditions that the operators must have in order to admit the restriction. In section 3, some such operators are found. We discuss some possible generalizations of the construction in the last section.

2. Centre of the quantum group and the restriction in roots of unity

As usual, let the R-matrix $R(u)$ denote the solution of the Yang–Baxter equation. We consider the simplest and well-known $R(u)$ matrix corresponding to the six-vertex model, whose elements can be written as

$$R^{\alpha\alpha}_{\alpha\alpha}(u) = \rho \sin(u+\eta) \qquad R^{\alpha\beta}_{\alpha\beta}(u) = \rho \sin u \qquad R^{\alpha\beta}_{\beta\alpha}(u) = \rho \sin \eta \tag{1}$$

where $\alpha, \beta = 1, 2$, $\alpha \neq \beta$ and η is the so-called anisotropy parameter.

The quantum group $\mathbf{A}$ connected with $R(u)$ is generated by $A(u), B(u), C(u), D(u)$, entries of the monodromy matrix $L(u)$, which satisfies

$$R_{12}(u-v)L_1(u)L_2(v) = L_2(v)L_1(u)R_{12}(u-v). \tag{2}$$

As shown by Tarasov [4], the centre of $\mathbf{A}$ in roots of unity, i.e. $\eta = \pi m/N$, is generated by the following elements of $\mathbf{A}$:

$$\langle A(u) \rangle = A(u)A(u+\eta)\cdots A(u+(N-1)\eta)$$
$$\langle B(u) \rangle = B(u)B(u+\eta)\cdots B(u+(N-1)\eta)$$
$$\langle C(u) \rangle = C(u)C(u+\eta)\cdots C(u+(N-1)\eta)$$
$$\langle D(u) \rangle = D(u)D(u+\eta)\cdots D(u+(N-1)\eta).$$

For convenience, we let $\langle B(u) \rangle$ denote the central element,

$$\langle B(u) \rangle = B(u)B_1(u)$$

where

$$B_1(u) = B(u+\eta)\cdots B(u+(N-1)\eta).$$

We now fix $V = C^2 \otimes \cdots \otimes C^2$ as the representation space of our quantum group. It is easy to see that for arbitrary v,

$$\langle B(v) \rangle = B(v)B_1(v) = 0$$

on this space. We can then define the state space of a restricted model as

$$W(v) = \text{Ker } B(v)/\text{Im } B_1(v).$$

In the limit $v \to \infty$, $B(v)$ coincides up to a scalar factor with X, one of the generators of $U_q(sl(2))$. As a result, $W(\infty)$ coincides with the space of the 'good' highest vectors of Pasqier–Saleur. It was shown in [2] that the Hamiltonian of the *XXZ* chain with OBC of special type

$$H_{XXZ} = \sum_{n=1}^{L-1} \left[\sigma_n^+ \sigma_{n+1}^- + \sigma_n^- \sigma_{n+1}^+ + \frac{\cos \eta}{2} \sigma_n^z \sigma_{n+1}^z + i \frac{\sin \eta}{2}\left(\sigma_n^z - \sigma_{n+1}^z\right) \right]$$

is invariant under the quantum algebra $U_q(sl(2))$. Here, $q = e^\eta$. Because of this, H_{XXZ} is properly defined on $W(\infty) = \text{Ker } X/\text{Im } X^{(N-1)}$.

In the thermodynamic limit, where $L \to \infty$, the spectrum of low-lying states coincides (in Cardy's sense) with $M(N - 1/N)$, one of the minimal models of CFT. In the next section, we show that the construction in [2] can be generalized to arbitrary values of the parameter v. The corresponding Hamiltonian is [3, 5]

$$H_{XXZ} = \sum_{n=1}^{L-1} \left[\sigma_n^+ \sigma_{n+1}^- + \sigma_n^- \sigma_{n+1}^+ + \frac{\cos \eta}{2} \sigma_n^z \sigma_{n+1}^z \right] + i \frac{\sin \eta}{2} \left(\cot v \sigma_1^z - \cot(v + \eta) \sigma_L^z \right). \tag{3}$$

We now find the sufficient conditions for any operator Q to be projected on $W(v)$. It is easy to see that sufficient conditions are that there exist some operators $\hat{Q}$ and $\hat{Q}_1$ for a given Q such that

$$B(v)Q = \hat{Q}B(v) \tag{4}$$

and

$$QB_1(v) = B_1(v)\hat{Q}_1. \tag{5}$$

Indeed, equation (4) guarantees that if a vector $\psi \in \mathrm{Ker}\, B(v)$, then the vector $Q\psi \in \mathrm{Ker}\, B(v)$. It follows from equation (5) that if the difference of two vectors ψ_1 and ψ_2 belong to $\mathrm{Im}\, B_1(v)$, i.e. if $\psi_1 - \psi_2 = B_1(v)\chi$, then the difference of $Q\psi_1$ and $Q\psi_2$ also belongs to $\mathrm{Im}\, B_1(v)$.

3. Sklyanin transfer matrix and other operators respecting the restriction in roots of unity

In [3], Sklyanin explained the integrability of the *XXZ* model with OBC of the form

$$H_{XXZ} = \sum_{n=1}^{L-1} \left[\sigma_n^+ \sigma_{n+1}^- + \sigma_n^- \sigma_{n+1}^+ + \frac{\cos \eta}{2} \sigma_n^z \sigma_{n+1}^z \right]$$
$$+ i \frac{\sin \eta}{2} \left(\cot(\xi_+ + \eta/2)\sigma_1^z + \cot(\xi_- - \eta/2)\sigma_L^z \right)$$

by constructing a special monodromy matrix and using it to diagonalize Hamiltonian (3) and the corresponding transfer matrix by means of the algebraic Bethe ansatz.

Let $K_+(u) = K(u + \eta/2, \xi_+)$ and $K_-(u) = K(u - \eta/2, \xi_-)$, where

$$K(u, \xi) = \begin{bmatrix} \sin(u + \xi) & 0 \\ 0 & -\sin(u - \xi) \end{bmatrix}.$$

Then $K_\pm$ satisfies the boundary Yang–Baxter equations [3, 6].

The Sklyanin monodromy matrix is defined [3] as

$$\Lambda(u) = \sigma^2 L^t(-u)\sigma^2 K_+(u)L(u)$$
$$= \begin{bmatrix} \Lambda_1^1 & \Lambda_1^2 \\ \Lambda_2^1 & \Lambda_2^2 \end{bmatrix}.$$

Using (2) and the boundary Yang–Baxter equation, we can prove that $\Lambda(u)$ satisfies the same equation as $K_+(u)$. It gives the commutation relations between Λ_j^i. The Sklyanin transfer matrix is defined as

$$T_S(u) = \mathrm{tr}\, \Lambda(u)K_-(u)$$
$$= \sin(u - \eta/2 + \xi_-)\Lambda_1^1 - \sin(u - \eta/2 - \xi_-)\Lambda_2^2.$$

Explicit expressions for the elements of the Sklyanin monodromy matrix are

$$\Lambda_1^1(u) = \sin(u + \eta/2 + \xi_+)A(u)D(-u) + \sin(u + \eta/2 - \xi_+)C(u)B(-u)$$

$$\Lambda_2^2(u) = -\sin(u + \eta/2 + \xi_+)B(u)C(-u) - \sin(u + \eta/2 - \xi_+)D(u)A(-u)$$

$$\Lambda_1^2(u) = \sin(u + \eta/2 + \xi_+)B(u)D(-u) + \sin(u + \eta/2 - \xi_+)D(u)B(-u)$$

$$\Lambda_2^1(u) = -\sin(u + \eta/2 + \xi_+)A(u)C(-u) - \sin(u + \eta/2 - \xi_+)C(u)A(-u).$$

Let $\xi_+ = v - \eta/2$ and $\xi_- = -v - \eta/2$. Then the following relations are satisfied:

$$B(v)T_S(u) = \widehat{T}_S(u)B(v) \tag{6}$$

where the explicit expression for the Sklyanin transfer matrix for these values ξ_{pm} is

$$T_S(u) = \sin(u - v - \eta)\sin(u + v)A(u)D(-u)$$

$$+ \sin(u - v - \eta)\sin(u - v + \eta)C(u)B(-u)$$

$$+ \sin(u + v)\sin(u + v)B(u)C(-u)$$

$$+ \sin(u + v)\sin(u - v + \eta)D(u)A(-u) \tag{7}$$

and

$$\widehat{T}_S(u) = \sin(u - v)\sin(u + v + \eta)A(u)D(-u) + \sin(u - v)\sin(u - v)C(u)B(-u)$$

$$+ \sin(u + v + \eta)\sin(u + v - \eta)B(u)C(-u)$$

$$+ \sin(u + v + \eta)\sin(u - v)D(u)A(-u).$$

We have

$$B(v)\Lambda_1^2(u) = \hat{\Lambda}_1^2(u)B(v) \tag{8}$$

where explicitly

$$\Lambda_1^2(u) = \sin(u + v)B(u)D(-u) + \sin(u - v + \eta)D(u)B(-u)$$

$$\hat{\Lambda}_1^2(u) = \sin(u + v + \eta)B(u)D(-u) + \sin(u - v)D(u)B(-u).$$

Equations (6) and (8) were verified by direct calculation.

The operators $T_S(u)$ and $\Lambda_1^2(u)$ thus satisfy the first condition, equation (4), for the restriction. We conjecture that they also satisfy the second one, equation (5), if q is a root of unity. This conjecture was explicitly verified by direct calculation for third-degree roots for the case $T_S(u)$ (but not for the case $\Lambda_1^2(u)$). It would be nice to find an elegant general proof.

The operators $T_S(u)$ and $\Lambda_1^2(u)$ depend on one parameter. There also exists a two-parameter family of operators satisfying (4) and (5).

By definition, let

$$T(x_{ij}; u_1, u_2) = x_{11}A(u_1)D(u_2) + x_{22}D(u_1)A(u_2) + x_{12}B(u_1)C(u_2) + x_{21}C(u_1)B(u_2).$$

Then

$$B(v)T(x_{ij}(v); u_1, u_2) = T(\hat{x}_{ij}(v); u_1, u_2)B(v) \tag{9}$$

where $T(x_{ij}(v); u_1, u_2)$ and $T(\hat{x}_{ij}(v); u_1, u_2)$ are obtained from $T(x_{ij}; u_1, u_2)$ by suitably substituting $x_{ij}(v)$ and $\hat{x}_{ij}(v)$ for x_{ij} and

$$x_{11}(v) = \sin(u_1 - \eta - v)\sin(u_2 - v)$$

$$x_{22}(v) = \sin(u_1 + \eta - v)\sin(u_2 - v)$$

$$x_{12}(v) = -\sin(u_2 - v)\sin(u_2 - v)$$

$$x_{21}(v) = -\sin(u_1 + \eta - v)\sin(u_1 - \eta - v)$$

$$\hat{x}_{11}(v) = \sin(u_1 - v)\sin(u_2 - \eta - v)$$

$$\hat{x}_{22}(v) = \sin(u_1 - v)\sin(u_2 + \eta - v)$$

$$\hat{x}_{12}(v) = -\sin(u_2 - \eta - v)\sin(u_2 + \eta - v)$$

$$\hat{x}_{21}(v) = -\sin(u_1 - v)\sin(u_1 - v).$$

We have

$$T(x'_{ij}(v); u_1, u_2)B(v + \eta)B(v + 2\eta) = B(v + \eta)B(v + 2\eta)T(\hat{x}'_{ij}(v); u_1, u_2) \tag{10}$$

where

$$x'_{11}(v) = \sin(u_1 - \eta - v)\sin(u_2 + 3\eta - v)$$
$$x'_{22}(v) = \sin(u_1 - 2\eta - v)\sin(u_2 - v)$$
$$x'_{12}(v) = -\sin(u_2 - v)\sin(u_2 - 3\eta - v)$$
$$x'_{21}(v) = -\sin(u_1 - \eta - v)\sin(u_1 - 2\eta - v)$$
$$\hat{x}'_{11}(v) = \sin(u_1 - 3\eta - v)\sin(u_2 - \eta - v)$$
$$\hat{x}'_{22}(v) = \sin(u_1 - v)\sin(u_2 - 2\eta - v)$$
$$\hat{x}'_{12}(v) = -\sin(u_2 - \eta - v)\sin(u_2 - 2\eta - v)$$
$$\hat{x}'_{21}(v) = -\sin(u_1 - v)\sin(u_1 - 3\eta - v).$$

If we require that the operators $T(x_{ij}(v); u_1, u_2)$ and $T(x'_{ij}(v); u_1, u_2)$ coincide, we can verify that this requirement is satisfied if $\eta = \pi/3$ or $\eta = 2\pi/3$. As discussed above, it follows that the two-parameter family of operators $T(x_{ij}(v); u_1, u_2)$ can be restricted on $W(v)$.

We conjecture that $T(x_{ij}(v); u_1, u_2)$ satisfies restriction conditions (4) and (5) if $\eta = m\pi/p$, where m and p are coprime integers.

It is easy to verify that the relation

$$T_S(u) = T(x_{ij}(v); u, -u)$$

holds, where $T_S(u)$ is Sklyanin transfer matrix in (7).

4. Discussion

It was shown in [7] that the Sklyanin transfer matrix for the Pasqier–Saleur case ($v \to \infty$) after the restriction satisfies the truncated system of fusion functional equations. This system defines the spectrum $M(p/p + 1)$. This statement can also be generalized to finite v. It is remarkable that the spectrum of states surviving after the restriction is independent of v [8]. This fact was discovered numerically in [5].

The explicit construction for additional central elements of the elliptic Yang–Baxter algebra in roots of unity was given in [9]. It would be interesting to generalize the approach in this paper to the elliptic case.

Another important problem is to generalize the Kitanine–Maillet–Terras construction [10] of the local operators of the *XXZ* model in terms of elements of the monodromy matrix to the restricted models. Namely, it would be interesting to build explicit operators that simultaneously respect the restriction and have mutual locality (i.e. commutativity). Constructing such operators would allow explicit formulae to be obtained for the correlation functions in the restricted models.

Acknowledgments

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